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Robust Output Regulation for Uncertain Linear Minimum Phase Systems under Unknown Control Direction ⋆

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Abstract: This paper investigates the problem of disturbance rejection for SISO uncertain linear minimum phase systems perturbed by an *unmeasurable* external disturbance under the framework of robust output regulation. The model parameters of systems in question are largely uncertain, including the control direction. In addition, the external disturbance is unstructured but bounded. Towards this end, a novel Unknown Input Observer (UIO)-based regulator is developed to reject the external disturbance, and a switching mechanism with a monitor function is designed to handle the control direction uncertainty. Notable features are that the unstructured external disturbance can be directly estimated and completely rejected by a sliding mode-based observer, and this new scheme can be applied for systems with non-unity relative degree under unknown control direction. The boundedness of closed-loop system and its asymptotic convergence properties are rigorously proved, which is verified by a numerical example.

Keywords: Disturbance rejection, uncertain systems, observer design, unknown control direction, switching mechanism

1. INTRODUCTION

The problem of tracking desired references while rejecting disturbances in the presence of model uncertainties is generically known as robust output regulation, which plays a central role through the history of control theory (Knobloch et al., 1993) and thus can be found in myriad engineering applications, including active rotor balancing (Zhou and Shi, 2001), active noise cancellation (Hansen et al., 2012) and active suspensions (Landau et al., 2005), etc. In practice, the reference signal is usually available, whereas the external disturbance to be rejected is more difficult to obtain, especially facing a time-varying uncertain operating environment. In this context, this work focuses on the more challenging task of disturbance rejection and considers an uncertain LTI SISO system described by:

$$\begin{aligned}\dot{x}(t) &= A(\mu)x(t) + B(\mu)[u(t) + d(t)], \quad x(0) = x_0 \in \mathcal{X}\\ y(t) &= C(\mu)x(t),\end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ represent the state, the input and the output of plant (1), respectively. The initial condition x_0 varies on a prescribed set $\mathcal{X} \subset \mathbb{R}^n$. $d(t)$ is the external disturbance of this system, which is unknown but assumed to be bounded.

The control objective is then posed as designing the control input $u(t)$ such that all signals are bounded in closed-loop system for any initial condition $x_0 \in \mathcal{X}$ and the output $y(t)$ is regulated asymptotically to zero without prior knowledge of the control direction.

Substantial studies on robust output regulation problem have been conducted since 1970s, to begin with the celebrated Internal Model (IM) principle (Francis and Wonham, 1976), i.e., the construction of a robust regulator that adaptively embeds the internal model of the exosystem, and later, the regulator design has been rapidly developed to more constructive and complicated methods in recent years, for instance, (Marino and Tomei, 2017, 2021; Wang et al., 2018, 2020; Jafari and Ioannou, 2016; Liang and Huang, 2021; Qian et al., 2021). However, in these works, the prior knowledge regarding the exosystem, i.e., frequency, is essential to reduplicate the internal model to solve the problem. Postulating the exosystem is unknown, Marino and Tomei (2011)'s adaptive learning regulator solved this problem under assumption that the system is minimum phase with known sign of high frequency gain. In Basturk and Krstic (2014), the problem was addressed when only state derivatives are measured. The above techniques, essentially, are suitable for systems with structured exosystems.

In the case when the exosystem is unstructured, instead of the aforementioned IM-based methods, the extended state observer-based Active Disturbance Rejection Con-

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trol (ADRC) and the sliding mode-based control provide alternatives to the disturbance rejection problem. The former suffers from the fact that stability analysis is nontrivial while the latter features a series of mature analysis technique but still requires some prior information of the plant. For instance, the well-known homogenous high-order sliding mode controller (Levant, 2003; Mercado-Uribe and Moreno, 2020) is proposed to solve the robust output regulation problem but the high-frequency gain is assumed to be positive and the nonlinear function of states is required to be bounded. Both assumptions limit its application. To relax these assumptions, Oliveira et al. (2015) proposed a sliding mode controller combined with the relay period switching, but only for the system with relative-degree-unity.

Based on above considerations, in this paper, we introduce a novel approach to address the robust output regulation problem for uncertain linear minimum phase systems under unknown control direction, in which the external disturbance is unmeasurable and unstructured. By means of appropriate change of suitable coordinate, the system in question can be reduced to a special normal form, characterized by the interconnection of so-called zero dynamics and a cascaded system. In this respect, this allows us to resort to a controller composed of sliding mode-based Unknown Input Observer (UIO) by Zhu et al. (2023) to reject the unknown and unstructured disturbance. An additional saturation function is employed to the controller such that the finite-time convergence property of the disturbance observer can always hold. To overcome the difficulty of unknown control direction, inspired by Oliveira et al. (2010), a switching scheme with monitor function is designed. We show that after finite switching, once the control input u is unsaturated, the output y is asymptotically regulated to zero and states of closed-loop system remain bounded.

The main novelties lie in the followings:

1) The proposed control method does not require the knowledge of the sign of high-frequency gain as prior and extends Oliveira et al. (2010)'s scheme to systems with arbitrary relative degree, which is the main contribution.
2) The controller is model-free and robust to the uncertain parameter set μ which allows to be arbitrarily large.
3) The unstructured external disturbance can be directly estimated and completely rejected by a sliding mode-based observer.

Notations: The following notations will be utilized in this paper: $\|\cdot\|$ represents the Euclidean norm of the matrice or vector; For any constant matrix $M \in \mathbb{R}^{m\times n}$, denote $M^+ = \max\{M, 0\}$ and $M^- = \max\{-M, 0\}$. Then obviously, we have $M = M^+ - M^-$ and $|M| = M^+ + M^-$, where $|M|$ stands for a $m \times n$ matrix formed by taking the absolute value of every element of M. A matrix or vector M $> (\geq, <, \leq)$ 0 means that all elements of matrix or vector are $>$ $(\geq, <, \leq)$ 0 respectively. In addition, a Metzler matrix is a square matrix whose off-diagonal components are all nonnegative. For any constant matrix $N \in \mathbb{R}^{n\times n}$, $N \succeq (\preceq)$ 0 means N is positive (negative) semi-definite, while $\lambda_{\max}(N)$ and $\lambda_{\min}(N)$ are denoted as the maximum and minimum eigenvalues of matrix N. The solution of discontinuous differential equations is understood in Filippov's definition (Filippov, 2013).

2. PROBLEM FORMULATION

In this section, we first reformulate the disturbance rejection problem such that a high-order sliding mode (HOSM)-based UIO can be employed to design a output feedback regulator to steer the output to zero.

Plant (1) is largely uncertain in the sense that the dependence of the matrices $A(\mu), B(\mu)$ and $C(\mu)$ on the unknown parameter vector $\mu \in \mathbb{R}^p$ is unknown but μ is assumed to range over a given compact set, $\mathcal{P} \subset \mathbb{R}^p$. However, the plant is assumed to be internally stable, robustly with respect to $\mu \in \mathcal{P}$, which is formally stated as follows:

Assumption 2.1. There exist constants $c_1, c_2 > 0$ such that the parameterized family $P_x(\mu) : \mathcal{P} \to \mathbb{R}^{n\times n}$ of solutions of the Lyapunov equation $P_x(\mu)A(\mu) + A^\top(\mu)P_x(\mu) = -\mathrm{I}_n$ satisfies $c_1\mathrm{I}_n \preceq P_x(\mu) \preceq c_2\mathrm{I}_n$ for all $\mu \in \mathcal{P}$.

Assumption 2.2. The plant in question is minimum phase with a known relative degree r.

Remark 2.1. Assumption 2.1 is not restrictive, since there are a large number of robust control methods such as "H_∞ control" (Zhou et al., 1996) and "high gain" stabilization (Teel and Praly, 1995) to achieve such internal stability property. Such hypothesis is assumed in rich literature (Astolfi et al., 2015; Marino and Tomei, 2017; Wang et al., 2020) focusing on disturbance rejection.

Observe that, $k_p(\mu) := C(\mu)A^{r-1}(\mu)B(\mu) \neq 0$ stands for the unknown high-frequency gain of plant (1). With an invertible coordinate change defined by $x \mapsto (\eta, \xi)$, the plant can be transformed into its normal form (Khalil, 1996, pp.512-514) described by:

$$\begin{aligned} \dot{\eta} &= A_o(\mu)\eta + B_o(\mu)y, \\ \dot{\xi} &= A_c\xi + B_c k_p(\mu)(u + d) + B_c\delta(x, \mu), \\ y &= C_c\xi, \end{aligned} \tag{2}$$

where $\eta \in \mathbb{R}^{n-r}$, $\xi := (\xi_1 \cdots \xi_r)^\top \in \mathbb{R}^r$, $\delta(x, \mu) := C(\mu)A^r(\mu)x$ and the pair $\{A_o, B_o\}$ are dependent on μ, besides the matrix $A_o(\mu)$ is Hurwitz due to the minimum phase condition while the pair $\{A_c, B_c, C_c\}$ is

$$A_c = \begin{pmatrix} \mathbf{0}_{(r-1)\times 1} & \mathbf{I}_{(r-1)\times(r-1)} \\ 0 & \mathbf{0}_{1\times(r-1)} \end{pmatrix},$$

$$B_c = \begin{pmatrix} \mathbf{0}_{1\times(r-1)} & 1 \end{pmatrix}^\top, \quad C_c = \begin{pmatrix} 1 & \mathbf{0}_{1\times(r-1)} \end{pmatrix}.$$

After adding and subtracting a term $\beta S_q(t)u$ to the right hand side of the differential equation of the last state ξ_r, system (2) can be rewritten in a compact form:

$$\begin{aligned} \dot{\eta} &= A_o(\mu)\eta + B_o(\mu)C_c\xi, \\ \dot{\xi} &= A_c\xi + B_c(\beta S_q(t)u + \varDelta(x, u, \mu, t)), \\ y &= C_c\xi, \end{aligned} \tag{3}$$

where β is a positive constant to be determined later, and $S_q(t)$ is a binary signal for which a switching scheme will be designed in the next section, to cycle through the set $\{-1, 1\}$. Note that, $\varDelta$ is the lumped uncertainty in the form of

$$\varDelta(x, u, \mu, t) := \delta(x, \mu) + (k_p(\mu) - \beta S_q)u + k_p(\mu)d. \tag{4}$$

In what follows, the uncertain parameter vector μ is ignored for neatness when no confusion is caused.

One non-conservative assumption needs to be made first for the state ξ of system (3) and the external input d:

Assumption 2.3. There exist two known constant vectors $\overline{\xi}(0)$ and $\underline{\xi}(0)$ conforming to $\underline{\xi}(0) \leq \xi(0) \leq \overline{\xi}(0)$ for all $x_0 \in \mathcal{X}$ and $\mu \in \mathcal{P}$. In addition, there exists a known positive constant $\overline{d}$ for disturbance d such that $|d(t)| \leq \overline{d}$.

Now, concerning the interconnected system (3), the observer-based regulator design problem is cast as:

Problem 2.1. Suppose Assumptions 2.1-2.3 hold, design a control law $u(t)$ for system (3) such that the trajectories of the closed-loop system are bounded w.r.t. any initial condition $x_0 \in \mathcal{X}$ and the output y of the plant asymptotically converges to zero as time goes to infinity. ◁

3. CONTROLLER DESIGN

The aim of this section is to design the control law u that solves Problem 2.1. For simplicity, the time argument has been omitted in the sequels unless necessary. Thanks to the controllability of the matrix pair $\{A_c, B_c\}$, we propose the following certainty-equivalent control law:

$$u = \overline{u}\ \mathrm{Sat}\left[\frac{-K\hat{\xi} - \hat{\Delta}}{\overline{u}\beta S_q}\right] \tag{5}$$

in which $\hat{\xi}$ and $\hat{\Delta}$ stand for the estimates for ξ and Δ, which will be given later by, respectively, a high-order sliding mode (HOSM)-based observer and a novel input reconstructor inspired by Zhu et al. (2023). The saturation function is defined as

$$\mathrm{Sat}\,[x] = \begin{cases} x, & \text{if } |x| \leq 1 \\ \mathrm{sign}(x), & \text{if } |x| > 1 \end{cases} \tag{6}$$

for any scalar variable x. Control gain K is chosen such that matrix $A_c - B_c K$ is Hurwitz and the selection of positive constant $\overline{u} \in \mathbb{R}$ will be discussed later.

3.1 High-order Sliding-mode (HOSM) Observer

In what follows, a HOSM differentiator with coefficients being designed under the saturated control input u (5) is proposed to achieve exact estimate of ξ in a finite time.

To this end, a necessary lemma is presented:

Lemma 3.1. There exist some positive constants k_1, k_2, k_3 dependent on μ such that the lumped disturbance is bounded, that is

$$|\Delta| \leq \overline{\Delta}$$

where we denote $\overline{\Delta} := k_1\overline{u} + k_2\overline{d} + k_3$ and $\overline{d}$ is the upper bound of disturbance d. ◁

The proof, found in Appendix A, is established employing Lyapunov function analysis. We conclude the following state-dependent upper bound for rth-order derivative of the output signal y satisfies

$$|\dot{\xi}_r| \leq k_4\overline{u} + k_2\overline{d} + k_3 \tag{7}$$

with $k_4 := k_1 + |\beta S_q|$.

In light of (7), a HOSM observer $\hat{\xi} := \left(\hat{\xi}_1 \ \cdots \ \hat{\xi}_r\right)^\top \in \mathbb{R}^r$ is proposed:

$$\begin{aligned} \dot{\hat{\xi}}_i &= \hat{\xi}_{i+1} + \hat{\nu}_i, \quad i = 1, \cdots, r-1 \\ \dot{\hat{\xi}}_r &= \beta S_q u + \hat{\nu}_r, \end{aligned} \tag{8}$$

where $\hat{\nu} := (\hat{\nu}_1 \ \cdots \ \hat{\nu}_r)^\top$ is generated by

$$\hat{\nu}_i = \tau_i \mathcal{L}^{\frac{1}{r+1-i}} \left|\hat{\nu}_{i-1}\right|^{\frac{r-i}{r+1-i}} \mathrm{sign}(\hat{\nu}_{i-1}), i = 1, \cdots, r \tag{9}$$

with $\hat{\nu}_0 = y - \hat{\xi}_1$, and positive tuning gains $\tau_i, \mathcal{L}$.

Now, the finite time convergence property can be established in the next lemma.

Lemma 3.2. Consider the HOSM differentiator (8), if the parameters τ_i are properly recursively chosen in accordance with Levant (2003), and static gain $\mathcal{L}$ satisfies

$$\mathcal{L} \geq k_4\overline{u} + k_2\overline{d} + k_3, \tag{10}$$

then, the following equations

$$\hat{\xi}_i = \xi_i, \quad i = 1, \cdots, r, \quad \forall t \geq T_1$$

hold for some finite time $T_1 > 0$. ◁

Proof. Subtracting the second equation in (3) from (8), the dynamics of the estimation error can be obtained as

$$\begin{aligned} \dot{\tilde{\xi}}_i &= \tilde{\xi}_{i+1} + \hat{\nu}_i, \quad i = 1, \cdots, r-1 \\ \dot{\tilde{\xi}}_r &\in \hat{\nu}_r + \left[-\overline{\Delta}, \overline{\Delta}\right], \end{aligned} \tag{11}$$

where $\left(\tilde{\xi}_1 \ \cdots \ \tilde{\xi}_r\right)^\top = \tilde{\xi} := \hat{\xi} - \xi$. It is shown in Levant (2003) that a sliding mode appears on the manifold $\tilde{\xi}_1 = \cdots = \tilde{\xi}_r = 0$ in a finite time by choosing the gains $\tau_i > 0$ properly and (10) is satisfied. Actually, we can always select a sufficiently large $\mathcal{L}$. In addition, the estimation error $\tilde{\xi}$ is bounded and monotonically decreasing. □

3.2 Interval Observer-based Estimator

To proceed, we will develop an interval observer-based estimator for Δ (4) that features a finite-time convergence property as well. Thanks to the observability of the matrix pair $\{A_c, C_c\}$, an interval observer for ξ-system (3) is constructed as:

$$\begin{aligned} \dot{\overline{\varsigma}} =& QA_cQ^{-1}\overline{\varsigma} + QB_c\beta S_q u + Q\gamma(y - C_cQ^{-1}\overline{\varsigma}) \\ &+ (QB_c)^+\overline{\Delta} - (QB_c)^-(-\overline{\Delta}), \\ \dot{\underline{\varsigma}} =& QA_cQ^{-1}\underline{\varsigma} + QB_c\beta S_q u + Q\gamma(y - C_cQ^{-1}\underline{\varsigma}) \\ &+ (QB_c)^+(-\overline{\Delta}) - (QB_c)^-\overline{\Delta}, \end{aligned} \tag{12}$$

in which $\overline{\Delta}$ is determined by Lemma 3.1 and the initial conditions are set as $\underline{\varsigma}(0) = Q^+\underline{\xi}(0) - Q^-\overline{\xi}(0)$ and $\overline{\varsigma}(0) = Q^+\overline{\xi}(0) - Q^-\underline{\xi}(0)$.

Proposition 1. (Zhu et al., 2023, Theorem 2) Under Assumption 2.3, states of system (12) verify $\underline{\varsigma}(t) \leq \varsigma(t) \leq \overline{\varsigma}(t)$ for all $t \geq 0$, if the gain vector γ together with matrix Q is chosen such that matrix $Q(A_c - \gamma C_c)Q^{-1}$ is not only *Hurwitz* but also *Metzler* [1]. ◁

Then, thanks to $\underline{\varsigma} \leq \varsigma \leq \overline{\varsigma}$ in Proposition 1 with the fact $\xi = Q^{-1}\varsigma$, the upper and lower boundary estimates of ξ can be calculated by

$$\begin{aligned} \overline{\xi} &= (Q^{-1})^+\overline{\varsigma} - (Q^{-1})^-\underline{\varsigma}, \\ \underline{\xi} &= (Q^{-1})^+\underline{\varsigma} - (Q^{-1})^-\overline{\varsigma}. \end{aligned} \tag{13}$$

[1] Readers are referred to Mazenc and Bernard (2011); TarekRaïssi et al. (2013) for concrete design procedures to derive Q, γ.

Now, based on the sliding mode observer (8) and the interval observer produced by (12)-(13), we are ready to employ an algebraic unknown input reconstruction method proposed by Zhu et al. (2023) to estimate the lumped uncertainty $\varDelta$ in (4). In virtue of (13), it is trivial to check that $\underline{\xi} \le \xi \le \overline{\xi}$ holds for all $t \ge 0$, which implies $\underline{\xi}_r \le \xi_r \le \overline{\xi}_r$, there must exist a time varying $\alpha_r(t)$ satisfying $0 \le \alpha_r(t) \le 1$ such that

$$\xi_r = \alpha_r(\overline{\xi}_r - \underline{\xi}_r) + \underline{\xi}_r. \tag{14}$$

Then differentiating (14) gives

$$\dot{\xi}_r = \dot{\alpha}_r(\overline{\xi}_r - \underline{\xi}_r) + \alpha_r(\dot{\overline{\xi}}_r - \dot{\underline{\xi}}_r) + \dot{\underline{\xi}}_r. \tag{15}$$

Denote $\widehat{\varsigma} := \overline{\varsigma} - \underline{\varsigma}$, $\widehat{\varDelta} := 2\overline{\varDelta}$. Note that, from the second equation of (3), we have $\dot{\xi}_r = \beta S_q u + \varDelta$, which together with (15), gives

$$\varDelta = \dot{\alpha}_r f_1(\widehat{\varsigma}) + \alpha_r f_2(\widehat{\varsigma}) + f_3(\overline{\varsigma}, \underline{\varsigma}) \tag{16}$$

in which

$$\begin{aligned} f_1(\widehat{\varsigma}) &= B_c^\top \left|Q^{-1}\right| \widehat{\varsigma}, \\ f_2(\widehat{\varsigma}) &= B_c^\top \left|Q^{-1}\right| \left[Q\left(A_c - \gamma C_c\right)Q^{-1}\widehat{\varsigma} + |QB_c|\,\widehat{\varDelta}\right], \\ f_3\left(\overline{\varsigma}, \underline{\varsigma}\right) &= B_c^\top (M_1\underline{\varsigma} - M_2\overline{\varsigma} + \gamma y + N_1(-\overline{\varDelta}) - N_2\overline{\varDelta}), \end{aligned} \tag{17}$$

and

$$\begin{aligned} M_1 &= (Q^{-1})^+ Q(A_c - \gamma C_c)Q^{-1}, \\ M_2 &= (Q^{-1})^- Q(A_c - \gamma C_c)Q^{-1}, \\ N_1 &= (Q^{-1})^+(QB_c)^+ + (Q^{-1})^-(QB_c)^-, \\ N_2 &= (Q^{-1})^+(QB_c)^- + (Q^{-1})^-(QB_c)^+. \end{aligned}$$

From (16), a re-constructor for the unknown input $\varDelta$ in (4) is obtained by

$$\hat{\varDelta} = \hat{\dot{\alpha}}_r f_1(\widehat{\varsigma}) + \hat{\alpha}_r f_2(\widehat{\varsigma}) + f_3(\overline{\varsigma}, \underline{\varsigma}) \tag{18}$$

where $\hat{\dot{\alpha}}_r$ and $\hat{\alpha}_r$ are the estimate of $\dot{\alpha}_r$ and α_r, respectively. Due to (14), $\hat{\alpha}_r$ can be computed by

$$\hat{\alpha}_r = \frac{\hat{\xi}_r - \underline{\xi}_r - \epsilon}{\overline{\xi}_r - \underline{\xi}_r + \epsilon} \tag{19}$$

with $\epsilon = 1$, if $\overline{\xi}_r = \underline{\xi}_r$; otherwise, $\epsilon = 0$.

In order to get the estimate of $\dot{\alpha}_r$ denoted by $\hat{\dot{\alpha}}_r$, we again resort to Levant (2003)'s second-order sliding model observer as follows:

$$\begin{aligned} &\dot{\rho}_1 = \iota_1, \iota_1 = -\kappa_1|\rho_1 - \hat{\alpha}_r|^{1/2}\mathrm{sign}(\rho_1 - \hat{\alpha}_r) + \rho_2, \\ &\dot{\rho}_2 = -\kappa_2\mathrm{sign}(\rho_2 - \iota_1) \end{aligned} \tag{20}$$

where ρ_2 is the exact estimate of $\dot{\alpha}_r$, with two positive scalar gains $\kappa_i > 0, i = 1, 2$ recursively chosen.

Proposition 2. Under Assumption 2.3, the estimator $\hat{\varDelta}$ in (18) that consists of (12), (17), (19) and (20) is able to provide for an exact estimate of lumped uncertainty $\varDelta$ in (4) within a finite time, that is, there exists a time instant $T_2 > 0$ such that the estimation error $\tilde{\varDelta} := \hat{\varDelta} - \varDelta = 0$, for all $t \ge T_2$. ◁

Proof. From (16) and (18), we can deduce that

$$\tilde{\varDelta} = \tilde{\dot{\alpha}}_r(t) f_1(\widehat{\varsigma}) + \tilde{\alpha}_r(t) f_2(\widehat{\varsigma})$$

with $\tilde{\dot{\alpha}}_r(t) := \hat{\dot{\alpha}}_r(t) - \dot{\alpha}_r(t)$ and $\tilde{\alpha}_r(t) := \hat{\alpha}_r - \alpha_r$. Thanks to Lemma 3.2, the finite time convergence of $\tilde{\xi}_r$ implies $\tilde{\alpha}_r = 0$ for all $t \ge T_1$. Since (20) features the same structure with (8), one can easily conclude that, after another period of time, say t_f, $\hat{\dot{\alpha}}_r$ must converge to $\dot{\alpha}_r$. Then, setting $T_2 := T_1 + t_f$, we have $\tilde{\varDelta} = 0$ for all $t \ge T_2$. Moreover, the estimation error $\tilde{\varDelta}$ is a bounded and monotonically decreasing signal during its transient period, thus completing the proof. □

3.3 Switching Scheme and Monitor Function Design

Before we discuss the switching scheme on S_q, a critical lemma is presented:

Lemma 3.3. If $S_q = \mathrm{sign}(k_p)$, there exists a constant β^* dependent on μ such that for all $\beta \ge \beta^*, \overline{u} \ge \overline{d}$, and $\forall t \ge T_2$, the unsaturated term $u_0 := \frac{-K\hat{\xi} - \hat{\varDelta}}{\beta S_q}$ is bounded as $|u_0| \le \overline{u}$, that is $u = u_0$. ◁

Proof can be found in Appendix B.

Remark 3.1. In fact, we can always select sufficiently large constants $\beta, \overline{u}$ independent of μ such that the inequalities $\beta \ge \beta^*, \overline{u} \ge \overline{d}$ hold. However, to make sure (10) satisfied, we shall select a larger constant $\mathcal{L}$, which results in a severe chattering effect of HOSM observers, and in turn, causes a trade-off between parameter design and control performance.

For the moment, let us suppose that the control direction is prior known (and set $S_q = \mathrm{sign}(k_p)$) and select $\beta \ge \beta^*, \overline{u} \ge \overline{d}$, the estimation errors $\tilde{\xi}$ and $\tilde{\varDelta}$ converge to zero in a finite time and during their transient period, signals $\tilde{\xi}$ and $\tilde{\varDelta}$ are bounded and monotonically decreasing from Lemma 3.2 and Proposition 2. Then, according to Lemma 3.3, there exists a time instant $T_{\mathrm{med}} \le T_2$ when u_0 is unsaturated, such that during $t \ge T_{\mathrm{med}}$, the closed-loop system of (3) in the form of

$$\begin{pmatrix}\dot{\eta} \\ \dot{\xi}\end{pmatrix} = \begin{pmatrix}A_o & B_oC_c \\ \mathbf{0} & A_c - B_cK\end{pmatrix}\begin{pmatrix}\eta \\ \xi\end{pmatrix} + \begin{pmatrix}\mathbf{0} & \mathbf{0} \\ -B_cK & -B_c\end{pmatrix}\begin{pmatrix}\tilde{\xi} \\ \tilde{\varDelta}\end{pmatrix} \tag{21}$$

is ISS with decaying input $\tilde{\xi}$ and $\tilde{\varDelta}$. Besides, the Lyapunov equation $(A_c - B_cK)^\top P_\xi + P_\xi(A_c - B_cK) = -\mathrm{I}_r$ has a symmetric and positive definite solution P_ξ.

Then, define the Lyapunov candidate function $V_2 = \sqrt{\xi^\top P_\xi \xi}$, its derivative along the solution of (21) satisfies

$$\begin{aligned} \dot{V}_2 \le &-\frac{1}{2\lambda_{\max}(P_\xi)}V_2 \\ &+ \frac{1}{\sqrt{\lambda_{\min}(P_\xi)}}(\|P_\xi B_cK\|\|\tilde{\xi}\| + \|P_\xi B_c\|\|\tilde{\varDelta}\|), \quad t \in [t_i, +\infty) \end{aligned}$$

for any $t_i \in [T_{\mathrm{med}}, +\infty)$. Hence, utilizing the comparison lemma (Khalil, 1996) and the fact $y = C_c\xi$, it follows

$$\begin{aligned} &|y(t)| \le \varPi(t), \\ &\varPi(t) := |y(t_i)|e^{-\frac{1}{2\lambda_{\max}(P_\xi)}(t - t_i)} + \pi(t), \quad t \in [t_i, +\infty) \end{aligned} \tag{22}$$

with $\pi(t) := \varPhi_1(\tilde{\xi}(0), t) + \varPhi_2(\tilde{\varDelta}(0), t)$ and $\varPhi_1, \varPhi_2 \in \mathcal{KL}$.

In this context, the major problem is that the sign of k_p in question is unknown, however, in light of the norm bound for y given in (22), we are able to construct the monitoring function $\varPsi_m$ inspired by Oliveira et al. (2010). Based on Lemma 3.3, the inequality (22) holds when S_q is chosen correct ($S_q = \mathrm{sign}(k_p)$), while y stays bounded when the selection of S_q is incorrect. It's natural to use Π in (22) as a benchmark to decide whether a switching of S_q is needed, i.e., the switching occurs only when (22) is violated. Nevertheless, since Π is not available for measurement we consider the following function, defined in the interval $[t_k, t_{k+1})$, to replace Π:

$$\Psi_k(t) = |y(t_k)|e^{-\lambda_m(t-t_k)} + \Theta(k)e^{-\lambda_c t} \quad (23)$$

where we select a sufficiently small positive constant λ_m and λ_c such that $\lambda_m \leq \frac{1}{2\lambda_{\max}(P_\xi)}$ holds. The switching time t_k sets the change of S_q, thus cycling through the set $\{-1, 1\}$ and $\Theta(k)$ is any positive monotonically increasing unbounded sequence. The monitoring function Ψ_m can thus be defined as

$$\Psi_m(t) := \Psi_k(t), \quad t \in [t_k, t_{k+1}) \subset [0, +\infty) . \quad (24)$$

Note that from (23) and (24), one can derive $|y(t_k)| < \Psi_k(t_k)$ at $t = t_k$. Hence, if the monitoring function $\Psi_m(t)$ is equal to $|y(t)|$, a switching will occur, that is

$$\begin{aligned} S_q(t_{k+1}) &= -S_q(t_k), \\ t_{k+1} &= \min\{t > t_k : |y(t)| = \Psi_k(t)\}, \end{aligned} \quad (25)$$

where t_k is the switching instant, $k \in \{0, 1, \cdots\}$ and $S_q(t_0) = 1$ with $t_0 = 0$. From (24), it is trivial to obtain the following inequality:

$$|y(t)| \leq \Psi_m(t), \quad \forall t \in [0, +\infty) . \quad (26)$$

4. STABILITY ANALYSIS

The main result is at present stated:

Theorem 3. Suppose Assumptions 2.1-2.3 hold, for plant (1) that can be transformed into its normal form (3), the controller (5) composed of HOSM estimators (8) and (18) and the monitoring function (23) and (24) fulfilling (10) and inequalities $\beta \geq \beta^*, \overline{u} \geq \overline{d}$, is able to stabilize the system and reject the unknown disturbance d in the sense that, for any initial condition $x_0 \in \mathcal{X}$, the trajectories of closed-loop system are all bounded and $\lim_{t\to\infty} y(t) = 0$, the problem of output regulation is solved. ◁

Proof. The proof is carried out in three parts:

1) The Monitoring Function Switching Stops:
Suppose by contradiction that S_q in controller (5) switches without stopping $\forall t \in [0, +\infty)$. Then, $\Theta(k)$ in (23) and (24) increases unboundedly as $k \to +\infty$. Thus, there exists a finite value κ such that for $k \geq \kappa$, we have: i) the term $\Theta(k)e^{-\lambda_c t}$ will upper bound $\pi(t)$ in (22), thus $\Psi_m(t) > \Pi(t), \forall t \in [t_k, t_{k+1})$, with Π in (22); and ii) S_q coincides with the sign of k_p, together with (10) and inequalities $\beta \geq \beta^*, \overline{u} \geq \overline{d}$ satisfied, the inequality (22) holds under Lemma 3.3, thus Π is a valid bound for $|y|$. Hence, no switching will occur after $t = t_\kappa$, i.e, $t_{\kappa+1} = +\infty$, which leads to a contradiction. Therefore, Ψ_m in (24) has to stop switching after some finite time $k = N$ and $t_N \in [0, +\infty)$.
2) Ultimate S_q Selected is such that $S_q = \mathrm{sign}(k_p)$:
Observe that if S_q is chosen correct, all trajectories of the system converge to zero, otherwise, for any initial condition the system trajectories do not converge to the origin. This is a contradiction, since if the switching stops, as aforementioned, the output must converge to the origin. Thus, the ultimate S_q selected is such that $S_q = \mathrm{sign}(k_p)$.
3) Closed-Loop Signal Boundedness and Exponential Convergence:
Since the switching stops and Ψ_m converges to zero exponentially, then, one concludes that y and states η, ξ will converge to zero at least exponentially. Reminding that u is bounded, then with Assumption 2.1, we can conclude that all closed-loop system signals are all bounded as well. □

5. SIMULATION

In this section, consider a second-order minimum phase system described by

$$G(s) = \begin{cases} -\dfrac{5}{s^2 + 3s + 10}, & \text{if } t \in [0, 30) \\ \dfrac{5}{s^2 + 3s + 10}, & \text{if } t \geq 30 \end{cases}$$

under the effect of a sinusoidal external disturbance

$$d(t) = \begin{cases} 2\sin(3t), & \text{if } t \in [0, 15) \\ 5\sin(t-1). & \text{if } t \geq 15 \end{cases}$$

All parameters of the system are unknown, including the control direction, only the output signal y is available to controller design. The input disturbance d is bounded by $\overline{d} = 5$. In this case, the proposed controller (5) with the switching mechanism is designed as follows:

The parameters utilized in the developed controller are set to $\overline{u} = 30$, $\beta = 1$, and the feedback gain vector K is designed to place the eigenvalues of $A_c - B_C K$ in $-1, -2$. The tunning gains in HOSM observer (8) and (20) are selected as $\mathcal{L} = 60, \tau_1 = 1.5, \tau_2 = 1.1$ and $\kappa_1 = 2.12, \kappa_2 = 2.2$ respectively. The parameters of the interval observer (12) are selected as $\gamma = (17\ 52)^\top$ and the transformation matrix $Q = \begin{pmatrix} 0.2500 & -0.0625 \\ 0.1538 & -0.0118 \end{pmatrix}$ employed in (12) is chosen so that

$$Q(A_c - \gamma C_c)Q^{-1} = \begin{pmatrix} -4 & 0 \\ 0 & -13 \end{pmatrix}$$

is a Hurwitz and Metzler matrix verifying the condition in Proposition 1. The initial condition of the plant is set as: $x(0) = (-1\ 2)^\top$, while the HOSM observers in (8) and (20) are initialized with $\hat{\xi}(0) = (0\ 0)^\top$, $\rho_1(0) = 0.5$, $\rho_2(0) = 0$. Thus, according to Assumption 2.3, the initial value of interval observer (12) can be chosen as $\overline{\xi} = (1\ 8)^\top$ and $\underline{\xi} = (-12\ 0)^\top$.

The monitor function Ψ_m is obtained from (23) and (24) with $\Theta(k) = k + 1$, $\lambda_c = \lambda_m = 0.3$.

Simulation results are shown in Figs. 1-3. At first 30 seconds, the switching signal S_q is initially set to 1 that is opposite to the sign of k_p. Thus the essential conditions in Lemma 3.3 are not satisfied, resulting in u saturated at the first 0.2 seconds. However, with the finite-time convergence of estimate error $\tilde{\xi}$ to zero in Fig. 3 and the fact that S_q goes through one necessary switching to reach the correct value in Fig. 2, control input u, once unsaturated, compensates for the unknown disturbance d and the output of plant (1) is regulated very quickly to zero in Fig. 1. After 30 seconds, a sudden change of sign of high frequency gain occurs. To be notable, similar to former results, after a finite time switching in Fig. 2, the external disturbance is completely rejected and the output converges to zero in a short time in Fig. 1, which explicitly shows the effectiveness of our control scheme.

As depicted in Fig. 1, another remarkable feature enjoyed by the proposed method is demonstrated via the transient behaviour around 15s, at which the frequency of external disturbance undergoes an abrupt change. From the magnified plot in Fig. 1, y goes back to zero after a small oscillation while the input u tracks the new disturbance instantaneously. Certainly, after the output being regulated to zero, one can obtain the frequency information applying many existing parameter estimation techniques to the control signal $u(t)$.

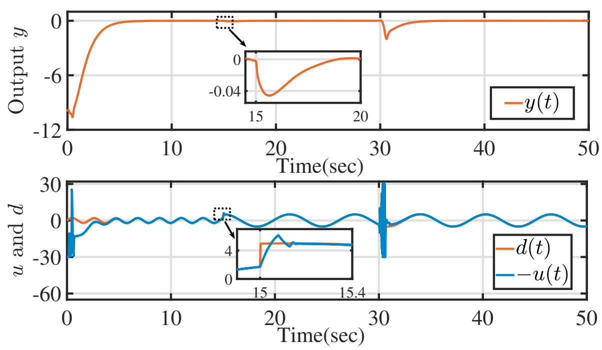

Fig. 1. Trajectories of inputs and output of system (1).

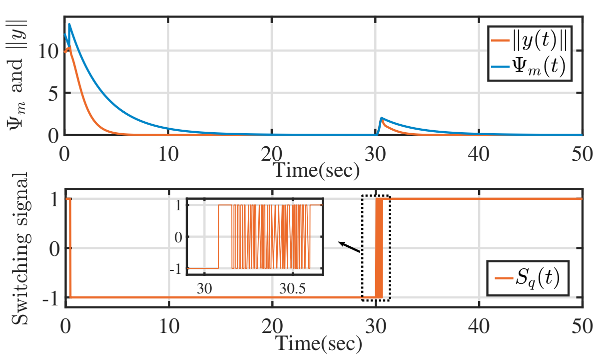

Fig. 2. Switching signal S_q and switching scheme with monitoring function (24).

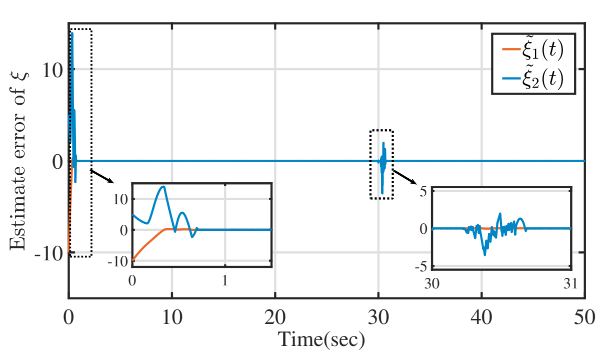

Fig. 3. Time history of estimate error $\tilde{\xi}$ (11).

6. CONCLUSIONS

In this paper, a novel UIO-based regulator is proposed to solve the output regulation problem for uncertain SISO linear minimum phase systems with arbitrary relative degree under unknown control direction. It is shown the resulting closed-loop system enjoys asymptotic stability and the unstructured disturbance can be completely rejected. The simulation results are consistent with the theoretical results and show the control objective is achieved by the proposed scheme. The future investigation should include the removal of minimum phase requirement and extension to MIMO systems, which is an intriguing challenge.

REFERENCES

Astolfi, D., Praly, L., and Marconi, L. (2015). Approximate regulation for nonlinear systems in presence of periodic disturbances. In *2015 54th IEEE Conference on Decision and Control (CDC)*, 7665–7670.

Basturk, H.I. and Krstic, M. (2014). State derivative feedback for adaptive cancellation of unmatched disturbances in unknown strict-feedback lti systems. *Automatica*, 50(10), 2539–2545.

Filippov, A.F. (2013). *Differential equations with discontinuous righthand sides: control systems*, volume 18. Springer Science & Business Media.

Francis, B. and Wonham, W.M. (1976). The Internal Model Principle of Control Theory. *Automatica*, 1098, 457–465.

Hansen, C., Snyder, S., Qiu, X., Brooks, L., and Moreau, D. (2012). *Active Control of Noise and Vibration, Second Edition*. Crc Pr I Llc.

Jafari, S. and Ioannou, P.A. (2016). Robust adaptive attenuation of unknown periodic disturbances in uncertain multi-input multi-output systems. *Automatica*, 70, 32–42.

Khalil, H.K. (1996). *Nonlinear Systems*. Prentice Hall.

Knobloch, H.W., Flockerzi, D., and Isidori, A. (1993). *Topics in Control Theory*. Birkhäuser Basel, Basel.

Landau, I.D., Constantinescu, A., and Rey, D. (2005). Adaptive narrow band disturbance rejection applied to an active suspension - An internal model principle approach. *Automatica*, 41(4), 563–574.

Levant, A. (2003). Higher-order sliding modes, differentiation and output-feedback control. *International Journal of Control*, 76(9-10), 924–941.

Liang, D. and Huang, J. (2021). Robust output regulation of linear systems by event-triggered dynamic output feedback control. *IEEE Transactions on Automatic Control*, 66(5), 2415–2422.

Marino, R. and Tomei, P. (2011). An adaptive learning regulator for uncertain minimum phase systems with undermodeled unknown exosystems. *Automatica*, 47(4), 739–747.

Marino, R. and Tomei, P. (2017). Hybrid Adaptive Multi-Sinusoidal Disturbance Cancellation. *IEEE Transactions on Automatic Control*, 62(8), 4023–4030.

Marino, R. and Tomei, P. (2021). Adaptive output regulation for minimum-phase systems with unknown relative degree. *Automatica*, 130, 109670.

Mazenc, F. and Bernard, O. (2011). Interval observers for linear time-invariant systems with disturbances. *Automatica*, 47(1), 140–147.

Mercado-Uribe, J.Á. and Moreno, J.A. (2020). Discontinuous integral action for arbitrary relative degree in sliding-mode control. *Automatica*, 118, 109018.

Oliveira, T.R., Peixoto, A.J., and Hsu, L. (2010). Sliding mode control of uncertain multivariable nonlinear systems with unknown control direction via switching and monitoring function. *IEEE Transactions on Automatic Control*, 55, 1028–1034.

Oliveira, T.R., Peixoto, A.J., and Hsu, L. (2015). Global tracking for a class of uncertain nonlinear systems with unknown sign-switching control direction by output feedback. *International Journal of Control*, 88, 1895–1910.

Qian, Y.Y., Liu, L., and Feng, G. (2021). Event-triggered robust output regulation of uncertain linear systems with unknown exosystems. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 51, 4139–4148.

TarekRaïssi, Efimov, D., and Zolghadri, A. (2013). Interval State Estimation for a Class of Nonlinear Systems. *IEEE Transactions On Automatic Control*, 23(2), 383–394.

Teel, A. and Praly, L. (1995). Tools for semiglobal stabilization by partial state and output feedback. *SIAM Journal on Control and Optimization*, 33(5), 1443–1488.

Wang, Y., Pin, G., Serrani, A., and Parisini, T. (2018). Switching-based sinusoidal disturbance rejection for uncertain stable linear systems. In *2018 Annual American Control Conference (ACC)*, 4502–4507.

Wang, Y., Pin, G., Serrani, A., and Parisini, T. (2020). Removing SPR-like conditions in adaptive feedforward control of uncertain systems. *IEEE Transactions on Automatic Control*, 65(6), 2309–2324.

Zhou, K., Doyle, J.C., and Glover, K. (1996). Robust and optimal control.

Zhou, S. and Shi, J. (2001). Active balancing and vibration control of rotating machinery: a survey. *Shock and Vibration Digest*, 33(5), 361–371.

Zhu, F., Fu, Y., and Dinh, T.N. (2023). Asymptotic convergence unknown input observer design via interval observer. *Automatica*, 147, 110744.

Appendix A. PROOF OF LEMMA 3.1

With the fact that $u \in \mathcal{L}_\infty$, we are able to prove Δ defined in (4) belongs to $\mathcal{L}_\infty$ as well.

Define the Lyapunov function $V_1 = x^\top P_x x$, the derivative $\dot{V}_1$ along the trajectory of (1) is

$$\begin{aligned}\dot{V}_1 &= -\|x\|^2 + 2x^\top P_x B(u+d)\\ &\leq -\frac{1}{c_2}V_1 + 2(\frac{V_1}{c_1})^{\frac{1}{2}}\|P_x B\|(\overline{u}+\overline{d})\end{aligned} \tag{A.1}$$

where $\overline{d}$ is the upper bound of disturbance d and here we exploit Rayleigh's inequality $c_1\|x\|^2 \leq V_1 \leq c_2\|x\|^2$.

Let $W_1 = V_1^{\frac{1}{2}}$, then $\dot{W}_1 = \frac{\dot{V}_1}{2\sqrt{V_1}}$. Dividing both side of (A.1) into $2W_1$, one obtains

$$\dot{W}_1 \leq -\frac{1}{2c_2}W_1 + \frac{1}{\sqrt{c_1}}\|P_x B\|(\overline{u}+\overline{d}). \tag{A.2}$$

By Comparison Principle (Khalil, 1996), it follows that

$$W_1 \leq e^{-\frac{1}{2c_2}t}W_1(0) - \frac{2c_2}{\sqrt{c_1}}\|P_x B\|(e^{-\frac{1}{2c_2}t}-1)(\overline{u}+\overline{d}).$$

Then, from Rayleigh's inequality, $W_1 \geq \sqrt{c_1}\|x\|$ and $W_1(0) \leq \sqrt{c_2}\|x(0)\|$, it follows that

$$\begin{aligned}\|x\| &\leq \frac{\sqrt{c_2}}{\sqrt{c_1}}e^{-\frac{1}{2c_2}t}\|x(0)\| - \frac{2c_2}{c_1}\|P_x B\|(e^{-\frac{1}{2c_2}t}-1)(\overline{u}+\overline{d})\\ &\leq \theta_1 e^{-\frac{1}{2c_2}t} + \theta_2(\overline{u}+\overline{d}),\end{aligned} \tag{A.3}$$

with

$$\theta_1 := \frac{\sqrt{c_2}}{\sqrt{c_1}}\|x(0)\| - \frac{2c_2}{c_1}\|P_x B\|(\overline{u}+\overline{d}), \quad \theta_2 := \frac{2c_2}{c_1}\|P_x B\|.$$

Recalling $\Delta = CA^r x + (k_p - \beta S_q)u + k_p d$, we are able to compute its bound

$$|\Delta| \leq \theta_3 e^{-\frac{1}{2c_2}t} + \theta_4\overline{u} + \theta_5\overline{d},$$

with $\theta_3 = \|CA^r\|\theta_1, \theta_4 = \|CA^r\|\theta_2 + |k_p - \beta S_q|, \theta_5 = \|CA^r\|\theta_2 + |k_p|$. In this respect, it is shown the lumped uncertainty Δ is norm-bounded by $k_1\overline{u} + k_2\overline{d} + k_3$, with $k_1 := \theta_4$, $k_2 := \theta_5$, $k_3 := \theta_3$. □

Appendix B. PROOF OF LEMMA 3.3

Since we set the saturation function for control input (5), the finite-time convergence property of Lemma 3.2 and Proposition 2 always holds, i.e., $\tilde{\xi} \to 0, \tilde{\Delta} \to 0$ after $t \geq T_2$. For simplicity of analysis, here we denote $\beta_q := \beta S_q$ and replace ξ and Δ in $u_0 = \frac{-K\hat{\xi}-\hat{\Delta}}{\beta_q}$.

First postulate the input of plant (1) is $\overline{u}+\overline{d}$. It's trivial to compute the solution of $x(t)$ is

$$x(t) = e^{At}x(0) + A^{-1}(e^{At} - \mathrm{I})B(\overline{u}+\overline{d}) \tag{B.1}$$

Due to $\xi_1 = Cx, \xi_2 = CAx, \cdots, \xi_r = CA^{r-1}x$, one obtains

$$\xi(t) = C_a e^{At}x(0) + C_a A^{-1}(e^{At} - \mathrm{I})B(\overline{u}+\overline{d}), \tag{B.2}$$

with $C_a = \begin{pmatrix} C^\top & \cdots & (CA^{r-1})^\top \end{pmatrix}^\top$.

Now from (B.1) and (B.2), it follows that

$$\begin{aligned}\frac{K\xi+\Delta}{\beta_q} &= \frac{K\xi}{\beta_q} + \frac{CA^r x}{\beta_q} + \frac{k_p - \beta_q}{\beta_q}\overline{u} + \frac{k_p\overline{d}}{\beta_q}\\ &= \theta_6 e^{At}x(0) + \theta_7 e^{At}B(\overline{u}+\overline{d}) - \frac{K}{\beta_q}C_a A^{-1}B(\overline{u}+\overline{d}) - \overline{u}\end{aligned}$$

where we make use of $k_p = CA^{r-1}B$, and denote $\theta_6 := \frac{KC_a + CA^r}{\beta_q}$ and $\theta_7 := \frac{KC_a A^{-1} + CA^{r-1}}{\beta_q}$.

with $CA^iB = 0, \forall i = 0, 1, \cdots, r-2$, one can derive

$$\frac{K}{\beta}C_a A^{-1}B = k_1\frac{CA^{-1}B}{\beta}, \tag{B.3}$$

where k_1 is the first element of control gain vector K, which is positive, due to $(A_c - B_c K)$ designed being Hurwitz. Moreover, $-CA^{-1}B$ is the DC gain of plant (1), equal to $k_p\theta_8(\mu)$, in which θ_8 is a positive constant depending on uncertain parameter set μ.

Thanks to (B.3), the signal u_0 in the form of

$$\theta_6 e^{At}x(0) + \theta_7 e^{At}B(\overline{u}+\overline{d}) + k_1\theta_8\frac{k_p}{\beta_q}(\overline{u}+\overline{d}) - \overline{u} \tag{B.4}$$

with two exponentially decaying terms will eventually norm-bounded by $\overline{u}$ if the following equality is satisfied

$$\mathrm{sign}(\beta_q) = \mathrm{sign}(k_p),$$

which indicates

$$S_q = \mathrm{sign}(k_p). \tag{B.5}$$

Moreover, if we design β such that $\beta \geq \beta^*$, for some positive constant $\beta^* := \frac{k_1\theta_8|k_p|}{2}$ and $\overline{u} \geq \overline{d}$. Then, $|u_0| \leq \overline{u}$, thus ending the proof. □